The $\hat{Q}$ operator for canonical quantum gravity

Yongge Ma[1]* and Yi Ling[2]†

[1] *FaMAF, Universidad Nacional de Cordoba, 5000, Cordoba, Argentina*

[2] *Center for Gravitational Physics and Geometry*
Department of Physics
The Pennsylvania State University
University Park, PA, USA 16802

February 7, 2008

Abstract

We study the properties of $\hat{Q}[\omega]$ operator on the kinematical Hilbert space $\mathcal{H}$ for canonical quantum gravity. Its complete spectrum with respect to the spin network basis is obtained. It turns out that $\hat{Q}[\omega]$ is diagonalized in this basis, and it is a well defined self-adjoint operator on $\mathcal{H}$. The same conclusions are also tenable on the $SU(2)$ gauge invariant Hilbert space with the gauge invariant spin network basis.

*E-mail: ygma@fis.uncor.edu
†E-mail: ling@phys.psu.edu

1 Introduction

It is well known that considerable progresses have been made in non-perturbative canonical quantum gravity in the light of Ashtekar's variables[1] and loop variables[2]. One of the most remarkable physical results is the evidence for a quantum discreteness of space at the Planck scale. This is due to the fact that certain operators corresponding to the measurement of area and volume have discrete spectra [3, 4, 5, 6]. Also, the kinematical Hilbert space, $\mathcal{H}$, of the theory has been rigorously defined by completing the space of all finite linear combinations of cylindrical functions in the norm induced by Haar measure [7, 8].

There is another geometrical operator $\hat{Q}[\omega]$ proposed in previous literatures [9, 10], which corresponds to the integrated norm of any smooth one form ω_a on the 3-manifold. While the operators of area and volume have been shown to be well defined self-adjoint operators on $\mathcal{H}$, the general properties of $\hat{Q}[\omega]$ are still unclear. We even do not know if it is well defined on $\mathcal{H}$. The obstacles are due to two facts: First, the result of $\hat{Q}[\omega]$ operating on a cylindrical function will involve integrals over edges of the graph on which the function defined, and hence it is no more a cylindrical function in general; Second, the current effective regularization technique of smearing the triads in 2-dimension[4, 11] could not be directly applied to the regularization of $\hat{Q}[\omega]$, whose classical expression involves the square of the triads while there is an integral over 3-dimensional manifold at last.

As $\hat{Q}[\omega]$ operator is rather convenient for constructing certain weave states in the study of the classical approximation of the quantum theory [9, 12], the present paper is devoted to study the properties of $\hat{Q}[\omega]$ on $\mathcal{H}$ in order to lay a foundation of its applications. To bypass the above mentioned obstacles, we will use a 3-dimensional smearing function for regularization. Then, instead of acting the regulated $\hat{Q}[\omega]$ on general cylindrical functions, we will operate it on spin network states which form a complete orthonormal basis in $\mathcal{H}$. It turns out that the operation gives a real discrete spectrum, which is in the same form as its eigenvalues on coloured loop states. Thus, $\hat{Q}[\omega]$ is a well defined symmetric operator in $\mathcal{H}$. A further discussion shows it is also self-adjoint.

We work in the real Ashtekar formalism defined over an oriented 3-manifold Σ [13]. The basic variables are real $SU(2)$ connections, A_a^i, as the configuration and the densitized(weight 1) triads, E_j^b, corresponding to the conjugate momentum. We use $a, b, \cdots$ for spatial indices and $i, j, \cdots$ for

internal $SU(2)$ indices. The basic variables satisfy

$$\{A_a^i(x), E_j^b(y)\} = G\delta_j^i\delta_a^b\delta^3(x,y), \tag{1}$$

where G is the usual gravitational constant.

2 $\hat{Q}$ operator and its regularization

The operator $\hat{Q}[\omega]$ is constructed to represent the classical quantity [9, 10]

$$Q[\omega] = \int d^3x\sqrt{E_i^a(x)\omega_a(x)E^{bi}(x)\omega_b(x)}, \tag{2}$$

where ω_a is any smooth 1-form on Σ which makes the integral meaningful and the integral is well defined since the integrand is a density of weight 1. If we know $Q[\omega]$ for all smooth ω_a, the triad E_i^a can be reconstructed up to local $SU(2)$ gauge transformations. Hence, the collection of $Q[\omega]$ provides a good coordinates system on the space of the triads fields.

Since E_i^a represents the conjugate momentum of the configuration variable A_a^i, the formal expression of the corresponding momentum operator would be some functional derivative with respect to A_a^i, i.e.,

$$\hat{E}_i^a(x) = -iG\hbar\frac{\delta}{\delta A_a^i(x)}. \tag{3}$$

This is an operator-valued distribution rather than a genuine operator, hence it has to be integrated against smearing functions in order to be well defined. Our aim is to construct a well defined operator $\hat{Q}[\omega]$ corresponding to the classical quantity $Q[\omega]$. We begin with a formal expression obtained by replacing E_i^a in Eq.(2) by the operator-valued distribution $\hat{E}_i^a$, and then regulate it by 3-dimensional smearing functions.

Let $f_\epsilon(x,y)$ be a 1-parameter family of fields on Σ which tends to $\delta(x,y)$ as ϵ tends to zero, such that $f_\epsilon(x,y)$ is a density of weight 1 in x and a function in y. We then define the smeared version of $E_i^a(x)\omega_a(x)$ as:

$$[E_i\omega]_f(x) := \int d^3y f_\epsilon(x,y)E_i^a(y)\omega_a(y). \tag{4}$$

Hence, $[E_i\omega]_f(x)$ tends to $E_i^a(x)\omega_a(x)$ as ϵ tends to zero. Then $Q[\omega]$ can be regulated as:

$$Q[\omega] = \lim_{\epsilon\to 0}\int d^3x\left([E_i\omega]_f(x)[E^i\omega]_f(x)\right)^{\frac{1}{2}}. \tag{5}$$

To go over to the quantum theory, we simply replace E_i^a by $\hat{E}_i^a$ and obtain

$$\hat{Q}[\omega] = \lim_{\epsilon\to 0}\int d^3x\left([\hat{E}_i\omega]_f(x)[\hat{E}^i\omega]_f(x)\right)^{\frac{1}{2}}, \tag{6}$$

where

$$[\hat{E}_i\omega]_f(x) := \int d^3y f_\epsilon(x,y)\hat{E}^a_i(y)\omega_a(y) = -iG\hbar \int d^3y f_\epsilon(x,y)\omega_a(y)\left(\frac{\delta}{\delta A^i_a(y)}\right). \tag{7}$$

By operating the regulated $\hat{Q}[\omega]$ on spin network states in next section, we will show that it is a well defined symmetric operator in the kinematical Hilbert space and admits self-adjoint extensions. For technical reasons, we attach the following concreteness to the smearing function f_ϵ for sufficiently small $\epsilon > 0$: (i) $f_\epsilon(x,y)$ is non-negative; and (ii) for any given y, $f_\epsilon(x,y)$ has compact support in x which is a 3-dimensional box, U_ϵ, of coordinate height ϵ^β, $1 < \beta < 2$, and square horizonal section, S_ϵ, of coordinate side ϵ, and with y as its centre. These conditions are in the same spirit as that in Ref.[14]. More concretely, $f_\epsilon(x,y)$ can be constructed as follows. Take any 1-dimensional non-negative function $\theta(x)$ of compact support $[-\frac{1}{2},\frac{1}{2}]$ on R such that $\int dx\theta(x) = 1$, and set

$$f_\epsilon(x,y) = \left(\frac{1}{\epsilon^{2+\beta}}\right)\theta\left(\frac{x_1-y_1}{\epsilon}\right)\theta\left(\frac{x_2-y_2}{\epsilon}\right)\theta\left(\frac{x_3-y_3}{\epsilon^\beta}\right). \tag{8}$$

3 Action of $\hat{Q}$ on spin network basis

3.1 Preliminaries

It has been shown that spin networks play a key role in non-perturbative quantum gravity [15, 8, 4]. Consider a graph Γ with n edges e_I, $I = 1,\cdots,n$, and m vertices v_α, $\alpha = 1,\cdots,m$, embedded in the 3-manifold Σ. To each e_I we assign a non-trivial irreducible spin j_I representation of $SU(2)$. This is called a colouring of the edge. Next, consider a vertex v_α, say a K-valent one, i.e., there are K edges $e_1,\cdots,e_K$ meeting at v_α. Let $\mathcal{H}_{j_1},\cdots,\mathcal{H}_{j_K}$ be the Hilbert spaces of the representations, $j_1,\cdots,j_K$, associated to the K edges. Consider the tensor product of these spaces $\mathcal{H}_{v_\alpha} = \mathcal{H}_{j_1}\otimes\cdots\otimes\mathcal{H}_{j_K}$, and fix, once and for all, an orthonormal basis, N_α, in $\mathcal{H}_{v_\alpha}$. This is called a colouring of the vertex. A (non-gauge invariant) spin network, S, is then defined as the embedded graph whose edges and vertices have been coloured.

The holonomy of the $SU(2)$ connection A^i_a along any edge e_I is an element of $SU(2)$ and can be expressed as:

$$h[A,e_I] = \mathcal{P}exp\int_{e_I} ds\dot{e}_I(s)A^i_a(e_I(s))\tau_i, \tag{9}$$

where $\mathcal{P}$ denotes path ordering and τ_i are the $SU(2)$ generators in the fundamental representation. The (non-gauge invariant) spin network state, $\Psi_S(A)$, based on S is defined as:

$$\Psi_S(A) = \bigotimes_{e_I\in\Gamma} j_I\left(h[e_I]\right)\cdot\bigotimes_{v_\alpha\in\Gamma} N_\alpha, \tag{10}$$

where $j_I\left(h[e_I]\right)$ is the representation matrix of the holonomy $h[e_I]$ in the spin j_I representation associated to the edge e_I, and the holonomy matrices are constructed with the vector N_α at each vertex v_α where the edges meet. By varying the graph, the colours of the edges, and the colours of the vertices, we obtain a family of spin network states. It turns out that these states form a complete orthonormal basis in the kinematical Hilbert space $\mathcal{H}$ [4, 11].

Since a $SU(2)$ gauge transformation acts on a spin network state simply by $SU(2)$ transforming the colouring of the vertices N_α, it is easy to recover the gauge invariant[1] spin network states by colouring each vertex with a $SU(2)$ invariant basis. These states form a complete orthonormal basis in the $SU(2)$ gauge invariant Hilbert space $\mathcal{H}_0$ [15, 8, 7].

It is obvious from Eqs. (3) and (9) that the action of $\hat{E}^a_i(x)$ on a holonomy $h[e_I]$ yields

$$\hat{E}^a_i(x)\circ h[e_I] = -il_p^2\int_{e_I} ds\dot{e}^a_I(s)\delta^3(x,e_I(s))h_I[1,s]\tau_i h_I[s,0], \tag{11}$$

where $l_p=\sqrt{G\hbar}$ is the Planck length.

3.2 Spectrum of $\hat{Q}$

We first apply the operator $[\hat{E}_i\omega]_f(x)$ defined by Eq.(7) to the spin network state Ψ_S,

$$\begin{aligned}
[\hat{E}_i\omega]_f(x)\circ\Psi_S(A) &= -il_p^2\sum_{I=1}^{n}\int d^3y f_\epsilon(x,y)\omega_a(y)\left[\frac{\delta}{\delta A^i_a(y)}j_I(h_I)_{lm}\right]\left(\frac{\partial\Psi_S}{\partial j_I(h_I)_{lm}}\right)\\
&= -il_p^2\sum_{I=1}^{n}\int d^3y f_\epsilon(x,y)\int_{e_I} dt\dot{e}^a_I(t)\omega_a(y)\delta^3(y,e_I(t))j_I\left(h_I[1,t]\tau_i h_I[t,0]\right)_{lm}\left(\frac{\partial\Psi_S}{\partial j_I(h_I)_{lm}}\right)\\
&= -il_p^2\sum_{I=1}^{n}\int_{e_I} dt\dot{e}^a_I(t)\omega_a(e_I(t))f_\epsilon(x,e_I(t))Tr\left[j_I\left(h_I[1,t]\tau_i h_I[t,0]\frac{\partial}{\partial h_I}\right)\right]\circ\Psi_S(A), \qquad (12)
\end{aligned}$$

where l and m are indices in $\mathcal{H}_{j_I}$ associated to e_I. Repeating the action of $[\hat{E}^i\omega]_f(x)$ on Eq.(12), we have

$$\begin{aligned}
[\hat{E}^i\omega]_f(x)[\hat{E}_i\omega]_f(x)\circ\Psi_S &= -l_p^4\sum_{I=1}^{n}\int_{e_I} dt\dot{e}^a_I(t)\omega_a(e_I(t))\sum_{J=1}^{n}\int_{e_J} ds\dot{e}^b_J(s)\omega_b(e_J(s))f_\epsilon(x,e_I(t))f_\epsilon(x,e_J(s))\\
&\quad Tr\left[j_J\left(h_J[1,s]\tau^i h_J[s,0]\frac{\partial}{\partial h_J}\right)\right]Tr\left[j_I\left(h_I[1,t]\tau_i h_I[t,0]\frac{\partial}{\partial h_I}\right)\right]\circ\Psi_S\\
&\quad -l_p^4\sum_{I=1}^{n}\int_{e_I} dt\dot{e}^a_I(t)\omega_a(e_I(t))f_\epsilon(x,e_I(t))\left(\int_t^1 ds\dot{e}^b_I(s)\omega_b(e_I(s))f_\epsilon(x,e_I(s))\right)
\end{aligned}$$

[1]The gauge invariance discussed in this paper is restricted to that of internal $SU(2)$, while the whole gauge invariance of a gravitational theory should also involve that of 4-dimensional diffeomorphism.

$$Tr\left[j_I\left(h_I[1,s]\tau^i h_I[s,t]\tau_i h_I[t,0]\frac{\partial}{\partial h_I}\right)\right] + \int_0^t ds\dot{e}^b_I(s)\omega_b(e_I(s))f_\epsilon(x,e_I(s))$$
$$Tr\left[j_I\left(h_I[1,t]\tau_i h_I[t,s]\tau^i h_I[s,0]\frac{\partial}{\partial h_I}\right)\right]\Bigg)\circ\Psi_S. \tag{13}$$

We denote respectively the first and second terms in the right hand side of Eq.(13) as A and B. Consider first the term A. Note that a spin network state can always be written as:

$$\Psi_S(A) = j_I\left(h[e_I]\right)_{lm}\Psi^{lm}_{S-e_I}(A), \tag{14}$$

where $\Psi^{lm}_{S-e_I}(A) = \frac{\partial\Psi_S}{\partial j_I(h_I)_{lm}}$ is independent of $h[e_I]$. Hence, we can choose ϵ sufficiently small, such that the term A vanishes unless the support U_ϵ of the smearing function f_ϵ contains a vertex v_α of the spin network as its centre. It then turns out

$$\begin{aligned}
A &= -l_p^4\sum_{I,J=1}^{n}\int_{e_I}dt\dot{e}^a_I(t)\omega_a(e_I(t))\int_{e_J}ds\dot{e}^b_J(s)\omega_b(e_J(s))[f_\epsilon(x,v_{IJ})]^2\\
&\quad Tr\left[j_J\left(h_J[1,s]\tau_i h_J[s,0]\frac{\partial}{\partial h_J}\right)\right]Tr\left[j_I\left(h_I[1,t]\tau_i h_I[t,0]\frac{\partial}{\partial h_I}\right)\right]\circ\Psi_S\\
&= -l_p^2\sum_{\alpha=1}^{m}[f_\epsilon(x,v_\alpha)]^2\sum_{I_\alpha,J_\alpha}\int_{e_I}dt\dot{e}^a_I(t)\omega_a(e_I(t))\int_{e_J}ds\dot{e}^b_J(s)\omega_b(e_J(s))\\
&\quad Tr\left[j_J\left(h_J[1,s]\tau_i h_J[s,0]\frac{\partial}{\partial h_J}\right)\right]Tr\left[j_I\left(h_I[1,t]\tau_i h_I[t,0]\frac{\partial}{\partial h_I}\right)\right]\circ\Psi_S.
\end{aligned} \tag{15}$$

To simplify technicalities, given a 1-form ω_a we choose f_ϵ such that at each vertex, ω_a is a normal covector of the horizonal section S_ϵ of U_ϵ, i.e., $\omega_a(v_\alpha) = |\omega(v_\alpha)|(dx_3)_a$. (Note that the special f_ϵ is chosen here in order to obtain a succinct expression of A, see Eq.(20), while the final result that A makes no contribution to the spectrum of $\hat{Q}$ is independent of this choice.) Since ϵ^β goes to zero faster than ϵ, for sufficiently small ϵ the edge e_I which meets the vertex v_α would cross the top or bottom of the box U_ϵ if it is not tangent to S_ϵ at v_α. Also, it follows from $\beta < 2$ that any edge tangential to S_ϵ at v_α exits U_ϵ from the side, irrespectively from its second (and higher) derivatives, and gives a vanishing contribution to A as ϵ goes to zero. Thus, if we first consider only the "outgoing" edges from the vertices, it turns out

$$\int_{e_I}dt\dot{e}^a_I(t)\omega_a(e_I(t))f_\epsilon(x,v_\alpha) = \frac{1}{2}\kappa_I\epsilon^\beta|\omega(v_\alpha)|f_\epsilon(x,v_\alpha) + O(\epsilon), \tag{16}$$

where

$$\kappa_I := \begin{cases} 0, & \text{if } e_I \text{ is tangent to } S_\epsilon\\ 1, & \text{if } e_I \text{ lies above } S_\epsilon\\ -1, & \text{if } e_I \text{ lies below } S_\epsilon\end{cases} \tag{17}$$

and hence Eq.(15) becomes

$$A_{out} = -\frac{1}{4} l_p^4 \sum_{\alpha=1}^{m} [f_\epsilon(x, v_\alpha)\epsilon^\beta |\omega(v_\alpha)|]^2 \sum_{I_\alpha, J_\alpha} [\kappa_I \kappa_J L_I^i L_J^i + O(\epsilon)] \circ \Psi_S, \tag{18}$$

where

$$L_I^i \circ \Psi_S := Tr\left[j_I \left(h[e_I]\tau^i \frac{\partial}{\partial h[e_I]}\right)\right] \circ \Psi_S. \tag{19}$$

Including the "incoming" edges to the vertices, the final expression of A reads

$$A = -\frac{1}{4} l_p^4 \epsilon^{2\beta} \sum_{\alpha=1}^{m} [f_\epsilon(x, v_\alpha)|\omega(v_\alpha)|]^2 \sum_{I_\alpha, J_\alpha} [\kappa_I \kappa_J X_I^i X_J^i + O(\epsilon)] \circ \Psi_S, \tag{20}$$

where

$$X_I^i \circ \Psi_S := \begin{cases} Tr\left[j_I \left(h[e_I]\tau^i \frac{\partial}{\partial h[e_I]}\right)\right] \circ \Psi_S, & \text{if } e_I \text{ is outgoing} \\ Tr\left[j_I \left(-\tau^i h[e_I] \frac{\partial}{\partial h[e_I]}\right)\right] \circ \Psi_S, & \text{if } e_I \text{ is incoming.} \end{cases} \tag{21}$$

Note that $\Delta_{S_\epsilon, v_\alpha} = \sum_{I_\alpha, J_\alpha} \kappa_I \kappa_J X_I^i X_J^i$ is the vertex operator associated with S_ϵ and v_α in arbitrary spin representations, which has been fully investigated [4, 16]. A discussion similar to that in Ref.[4] leads that the spin network state Ψ_S is an eigenvector of $-\Delta_{S_\epsilon, v_\alpha}$ with eigenvalue:

$$\lambda_{S_\epsilon, v_\alpha} = 2j^{(d)}(j^{(d)}+1) + 2j^{(u)}(j^{(u)}+1) - j^{(d+u)}(j^{(d+u)}+1), \tag{22}$$

where $j^{(d)}$, $j^{(u)}$ and $j^{(d+u)}$ are half integers subject to the condition
$j^{(d+u)} \in \{|j^{(d)} - j^{(u)}|, |j^{(d)} - j^{(u)}| + 1, \cdots, j^{(d)} + j^{(u)}\}$.

Now consider the second term B. For small ϵ, $f_\epsilon(x, e_I(t)) f_\epsilon(x, e_I(s))$ is non-zero only for the parameters satisfying $t = s + O(\epsilon)$, where we have

$$\begin{aligned} Tr\left[j_I \left(h_I[1,t]\tau_i h_I[t,s]\tau^i h_I[s,0]\frac{\partial}{\partial h_I}\right)\right] \circ \Psi_S &= \left(Tr\left[j_I \left(h_I[1,t]\tau_i \tau^i h_I[t,0]\frac{\partial}{\partial h_I}\right)\right] + O(\epsilon)\right) \circ \Psi_S \\ &= \left(-j_I(j_I+1) Tr\left[j_I \left(h_I[e_I]\frac{\partial}{\partial h[e_I]}\right)\right] + O(\epsilon)\right) \circ \Psi_S \\ &= -[j_I(j_I+1) + O(\epsilon)]\Psi_S, \end{aligned} \tag{23}$$

here $-j_I(\tau_i \tau^i) = j_I(j_I+1)$ is the Casimir operator of $SU(2)$. Substituting Eq.(23) into B, we obtain

$$B = l_p^4 \sum_{I=1}^{n} \left[\int_{e_I} dt \dot{e}_I^a(t) \omega_a(e_I(t)) f_\epsilon(x, e_I(t))\right]^2 [j_I(j_I+1) + O(\epsilon)]\Psi_S. \tag{24}$$

It then follows from Eqs. (20) and (24) that

$$[\hat{E}^i\omega]_f(x)[\hat{E}_i\omega]_f(x)\circ\Psi_S = \frac{1}{4}\epsilon^{2\beta}l_p^4\sum_{\alpha=1}^{m}[f_\epsilon(x,v_\alpha)|\omega(v_\alpha)|]^2(\lambda_{S_\epsilon,v_\alpha}+O(\epsilon))\Psi_S$$
$$+l_p^4\sum_{I=1}^{n}\left[\int_{e_I}dt\dot{e}_I^a(t)\omega_a(e_I(t))f_\epsilon(x,e_I(t))\right]^2[j_I(j_I+1)+O(\epsilon)]\Psi_S, \qquad (25)$$

which implies that $[\hat{E}^i\omega]_f(x)[\hat{E}_i\omega]_f(x)$ is a well defined non-negative operator and hence has a well defined square-root. Since we have chosen ϵ to be sufficiently small, for any given $x\in\Sigma$, $f_\epsilon(x,v_\alpha)$ is non-zero for at most one vertex, and $f_\epsilon(x,e_I(t))$ is non-zero for at most a piece of one edge where its vertices are not included. Therefore we can take the sum over v_α and e_I out side the square root and obtain

$$\left([\hat{E}^i\omega]_f(x)[\hat{E}_i\omega]_f(x)\right)^{\frac{1}{2}}\circ\Psi_S = \frac{1}{2}\epsilon^{\beta}l_p^2\sum_{\alpha=1}^{m}f_\epsilon(x,v_\alpha)|\omega(v_\alpha)|(\lambda_{S_\epsilon,v_\alpha}+O(\epsilon))^{\frac{1}{2}}\Psi_S$$
$$+l_p^2\sum_{I=1}^{n}\left|\int_{e_I}dt\dot{e}_I^a(t)\omega_a(e_I(t))f_\epsilon(x,e_I(t))\right|[j_I(j_I+1)+O(\epsilon)]^{\frac{1}{2}}\Psi_S. \qquad (26)$$

Now we can remove the regulator. Taking the limit $\epsilon\to 0$ and integrating over Σ, the first term in the right hand side of Eq.(26) vanishes due to the factor ϵ^β. We thus conclude that the action of $\hat{Q}[\omega]$ on spin network states yields

$$\hat{Q}[\omega]\circ\Psi_S(A) = l_p^2\sum_{I=1}^{n}\left[\int_{e_I}dt|\dot{e}_I^a(t)\omega_a(e_I(t))|\sqrt{j_I(j_I+1)}\right]\Psi_S(A). \qquad (27)$$

Therefore, spin network states are also eigenvectors of $\hat{Q}[\omega]$. The complete spectrum of $\hat{Q}[\omega]$ with respect to the spin network basis in the Hilbert space $\mathcal{H}$ is obtained.

4 Discussions

The general properties of $\hat{Q}[\omega]$ operator are implied by Eq.(27). In contrast to the volume operator which acts only on vertices [5, 6], $\hat{Q}[\omega]$ acts only on edges of spin networks. Hence, the spin network states based on a same graph with same colouring of the edges are all degenerate with respect to this operator. As a result, the action of $\hat{Q}[\omega]$ on the gauge invariant spin network states gives the same result as Eq.(27). In this sense, the spectrum of $\hat{Q}[\omega]$ respects the physically relevant states in $\mathcal{H}$.

There are alternative approaches to regulate $\hat{Q}[\omega]$ and calculate its spectrum. One could also apply the blocking regularization technique of Ref.[14], then express $\hat{Q}[\omega]$ by the loop operator $\mathcal{T}^{ab}$

up to $O(\epsilon)$, whose action on spin network states is obtained from the recoupling theory [5]. It is not difficult to check that this approach will give the same result as we have obtained. By restricting the support of the regulator, our approach reveals the inherent relation of the two approaches.

We have shown that $\hat{Q}[\omega]$ is diagonalized in the spin network basis with real eigenvalues, hence it is a well defined symmetric operator in the kinematical Hilbert space $\mathcal{H}$. Moreover, it is obvious from Eqs. (6) and (7) that the expression of $\hat{Q}[\omega]$ is purely real, and hence it commutes with the complex conjugation. Therefore, it follows from Von Neumann's theorem[17] that $\hat{Q}[\omega]$ admits self-adjoint extensions on $\mathcal{H}$. The same reasons lead that $\hat{Q}[\omega]$ is also self-adjoint on the gauge invariant Hilbert space $\mathcal{H}_0$.

The discrete spectrum of $\hat{Q}[\omega]$ shows a quantum discreteness of the space at the Planck scale, corresponding to the measurement of the integrated norm of any smooth one forms.

Acknowledgements

Y. Ma would like to thank discussions with Profs. C. N. Kozameh, O. M. Moreschi, and especially O. A. Reula for his enlightening comments and suggestions, and acknowledge support from FONCYT BID 802/OC-AR PICT: 00223. Y. Ling is grateful to Abhay Ashtekar for discussion, and was supported by the NSF through grant PHY95-14240, a gift from the Jesse Phillips Foundation, and a Braddock fellowship from the Department of Physics at Pennsylvania State University.

References

[1] A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986).

[2] C. Rovelli and L. Smolin, Nucl. Phys. B**331**, 80 (1990).

[3] C. Rovelli and L. Smolin, Nucl. Phys. B**442**, 593 (1995).

[4] A. Ashtekar and J. Lewandowski, Class. Quant. Grav. **14**, A43 (1997).

[5] R. DePietri and C. Rovelli, Phys. Rev. D**54**, 2664 (1996).

[6] A. Ashtekar and J. Lewandowski, Adv. Theor. Math. Phys. **1**, 388 (1998).

[7] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourao, and T. Thiemann, J. Math. Phys. **36**, 6456 (1995).

[8] J.C. Baez, in *The Interface of Knots and Physics*, L. Kauffman, editor, (Providence, Rhode Island, 1996), gr-qc/9504036.

[9] A. Ashtekar, C. Rovelli, and L. Smolin, Phys. Rev. Lett. **69**, 237 (1992).

[10] L. Smolin, in *The Proceeding of the XXII Gift International Seminar on Theoretical Physics*, Catalonia, Spain, June 1991, (World Scientific, Singapore, 1992), hep-th/9202022.

[11] M. Gaul and C. Rovelli, in *The Proceeding of the XXXVth Karpacz Winter School on Theoretical Physics*, Polanica, Poland, 2-12 Feb. 1999, gr-qc/9910079.

[12] Y. Ma and Y. Ling, Phys. Rev. D**62**, 064030 (2000).

[13] F. Barbero, Phys. Rev. D**51**, 5507 (1995).

[14] S. Fritelli, L. Lehner, and C. Rovelli, Class. Quant. Grav. **13**, 2921 (1996).

[15] C. Rovelli and L. Smolin, Phys. Rev. D**52**, 5743 (1995).

[16] A. Ashtekar and J. Lewandowski, J. Geo. Phys. **17**, 191 (1995).

[17] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness*, Page 143, (Acdamic Press, 1975).